

# BF gravity with Immirzi parameter and matter fields

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We perform the coupling of the scalar, Maxwell, and Yang-Mills fields as well as the cosmological constant to  $BF$  gravity with Immirzi parameter. The proposed action principles employ auxiliary fields in order to keep a polynomial dependence on the  $B$  fields. By handling the equations of motion for the  $B$  field and for the auxiliary fields, these latter can be expressed in terms of the physical fields and by substituting these expressions into the original action principles we recover the first-order (Holst) and second-order actions for gravity coupled to the physical matter fields. We consider these results a relevant step towards the understanding of the coupling of matter fields to gravity in the theoretical framework of  $BF$  theory.

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## I. INTRODUCTION

The research in quantum gravity led by its two main branches (loop quantum gravity [1] and spin foam models for gravity [2]) has recently motivated the study of the classical descriptions for general relativity and theories related to it, particularly the formulations of gravity as a constrained  $BF$  theory. Just to mention some of them, Cartan's equations in the framework of  $BF$  theories are analyzed in Refs. 3 and 4 while the relationship of general relativity to the Husain-Kuchar model in the framework of  $BF$  theory is analyzed in Refs. 5–7.

It is possible to say that loop quantum gravity and spin foam models for gravity are, in a certain sense, inspired in the Plebański's work [8]. As is well-known, in the mid-70's of the twentieth century Plebański wrote the equations of motion for four-dimensional general relativity in such a way that the fundamental variables for describing the gravitational field are two-form fields, a connection one-form, and some Lagrange multipliers. The geometry of the spacetime is built up from these fundamental blocks. The Plebański action is a  $BF$  theory supplemented with constraints on some of the fields involved. In order to bring tetrads into the formulation, the two-forms  $B$ 's are eliminated by solving the constraints on them, which implies that the  $B$ 's can be expressed in terms of tetrad fields, and by inserting back this into Plebański's action, it becomes the self-dual action for general relativity [9, 10]. At the beginning of the 1990's Plebański's formulation was extended in order to include the coupling of matter fields [11].

Following the same idea used by Plebański, at the beginning of this century an action principle for real general relativity including the Immirzi parameter was introduced by Capovilla, Montesinos, Prieto and Rojas in Ref. 12 (CMPR action principle). This action is very close to the Plebański action because it is given in terms of two-forms, a connection one-form, and some Lagrange multipliers but now it includes an arbitrary value of the Immirzi parameter. This is not the unique formulation for real  $BF$  gravity, there also exist another one used by Engle, Pereira, and Rovelli in Ref. 13. These two formulations can be related by doing a transformation between the set of fields involved in one of them and the set of fields involved in the other. A detailed analysis of this transformation can be found in Ref. 14 (see also Ref. 15). The transformation allows us to translate any analysis made in one formulation to the other one. In particular, the coupling of matter fields to  $BF$  gravity with Immirzi parameter can be done in any of these two approaches, and using the transformation such a coupling can be done in the other framework. As an example of this theoretical framework, the inclusion of the cosmological constant to the action principle used in Ref. 13 was obtained in Ref. 14 from the coupling of the cosmological constant to the CMPR action principle done in Ref. 16.

Following this trend of ideas, in this paper we focus our attention in the coupling of the cosmological constant, the scalar, Yang-Mills, and Maxwell fields to gravity in the  $BF$  framework. Our starting point is the action principle used in Ref. 13 but, as mentioned above, once the coupling is done, it is also possible to obtain such a coupling in the CMPR action principle. For each one of the couplings presented in this paper, it can be shown that Einstein's equations and the equations of motion corresponding to the involved matter field follow immediately from the proposed action principle. This will be explicitly shown for the coupling of the cosmological constant while for the coupling of the

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other matter fields, the proposed action principles will be rewritten in terms of tetrad fields by solving the constraints on the two-forms and the equations of motion for the auxiliary fields. Following this approach, we will obtain Holst's action principle for gravity coupled with the correspondent matter field. Owing to this equivalence, it is pretty obvious that the equations of motion that follow from the proposed action principles are equivalent to Einstein's equations coupled to matter fields plus the equations of motion for the matter fields themselves. The material reported in this paper is part of the work presented in Ref. 17.

The starting point for the coupling of matter fields to gravity is the action principle for general relativity given by

$$S_{GR}[B, A, \Phi, \mu] = \int_{\mathcal{M}^4} \left[ \left( B^{IJ} + \frac{1}{\gamma} *B^{IJ} \right) \wedge F_{IJ}[A] - \frac{1}{2} \Phi_{IJKL} B^{IJ} \wedge B^{KL} - \mu \Phi_{IJKL} \varepsilon^{IJKL} \right], \quad (1)$$

where  $F^I{}_J[A] = dA^I{}_J + A^I{}_K \wedge A^K{}_J$  is the curvature of the  $SO(4)$  or  $SO(3,1)$  connection one-form  $A^{IJ} = -A^{JI}$  and the  $B^{IJ}$  are six two-forms because of the property  $B^{IJ} = -B^{JI}$ ; the internal indices  $I, J, K, \dots$  take on the values 0,1,2,3 and they are raised and lowered with the metric  $(\eta_{IJ}) = (\sigma, 1, 1, 1)$  where  $\sigma = -1$  for Lorentzian and  $+1$  for Euclidean signatures, respectively. We define  $*B^{IJ} = \frac{1}{2} \varepsilon^{IJ}{}_{KL} B^{KL}$ ,  $\mu$  is a 4-form and  $\Phi_{IJKL}$  has the usual symmetries  $\Phi_{IJKL} = \Phi_{KLIJ}$ ,  $\Phi_{IJKL} = -\Phi_{JIKL}$ , and  $\Phi_{IJKL} = -\Phi_{IJLK}$ . See Refs. 18 and 19 for the Lorentz-covariant Hamiltonian analysis of the action (1).

## II. COUPLING THE COSMOLOGICAL CONSTANT

The coupling of the cosmological constant to the CMPR action principle has been studied in Refs. 16 and 20. The coupling of the cosmological constant presented here is done following our ideas reported in Refs. 16 and 21. In order to introduce the cosmological constant into the action principle (1), we propose the following action principle

$$S[B, A, \Phi, \mu] = S_{GR}[B, A, \Phi, \mu] + \int_{\mathcal{M}^4} (\mu\lambda + l_1 B_{IJ} \wedge B^{IJ} + l_2 B_{IJ} \wedge *B^{IJ}), \quad (2)$$

where  $\lambda$ ,  $l_1$ , and  $l_2$  are constants whose relationship with the cosmological constant  $\Lambda$  will be analyzed below. The variation of this action gives the equations of motion

$$\delta B : F_{IJ}[A] + \frac{1}{\gamma} *F_{IJ} - \Phi_{IJKL} B^{KL} + 2l_1 B_{IJ} + 2l_2 *B_{IJ} = 0, \quad (3a)$$

$$\delta A : DB^{IJ} + \frac{1}{\gamma} D*B^{IJ} = 0, \quad (3b)$$

$$\delta \Phi : B^{IJ} \wedge B^{KL} + 2\mu \varepsilon^{IJKL} = 0, \quad (3c)$$

$$\delta \mu : \Phi_{IJKL} \varepsilon^{IJKL} = \lambda. \quad (3d)$$

In what follows it will be shown that Eqs. (3) imply that the Einstein's equations with cosmological constant given by

$$*F_{IJKL} + *F_{IKLJ} + *F_{ILJK} = \Lambda \varepsilon_{IJKL}, \quad (4)$$

with  $F_{IJ} = \frac{1}{2} F_{IJKL} e^K \wedge e^L$ , are completely satisfied.

We are going to begin the analysis of the equations of motion (3). It can be easily seen that Eq. (3a) can be written as<sup>1</sup>

$$F_{IJ} = \frac{\gamma^2}{\gamma^2 - \sigma} \left[ \left( \Phi_{IJKL} - \frac{1}{\gamma} * \Phi_{IJKL} \right) B^{KL} + 2 \left( \frac{l_2 \sigma}{\gamma} - l_1 \right) B_{IJ} + 2 \left( \frac{l_1}{\gamma} - l_2 \right) *B_{IJ} \right], \quad (5)$$

provided that  $\gamma^2 \neq \sigma$ . Under the same restriction, Eq. (3b) reduces to

$$DB^{IJ} = 0. \quad (6)$$

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<sup>1</sup> Note that the fact that Eq. (5) holds for  $\gamma^2 \neq \sigma$  is a feature already present in the action (1) for pure gravity and does not come from the coupling of the cosmological constant itself. Furthermore, the action (1) does *not* reduce to the Plebański action [8] for the choices of  $\gamma^2 = \sigma$  as it is explained in detail in Ref. 17 where, by the way, an action principle having the right self-dual limits is reported. In spite of this property, the action (1) has been subject of study by Engle, Pereira, and Rovelli in Ref. 13 and more recently by Perez in the last paper of Ref. 2.

On the other hand, Eqs. (3c) imply that there exist two independent solutions for the two-forms  $B^{IJ}$  given by

$$B^{IJ} = \kappa_1 * (e^I \wedge e^J), \quad (7a)$$

$$B^{IJ} = \kappa_2 e^I \wedge e^J, \quad (7b)$$

where  $\kappa_1$  and  $\kappa_2$  are constants. For any of these solutions, Eq. (6) reduces to  $De^I = 0$ . This means that  $A^I{}_J$  is the spin-connection. Therefore, the curvature  $F^I{}_J$  must satisfy the Bianchi identities without torsion given by

$$F_{IJKL} + F_{IKLJ} + F_{ILJK} = 0. \quad (8)$$

In summary, Eqs. (3b) and (3c) imply that  $A^I{}_J$  is the spin-connection,  $F^I{}_J$  satisfies Bianchi identities without torsion (8), and that the two-form  $B$  can be written as given in (7). Moreover, due to the fact that Eq. (5) expresses the curvature  $F_{IJ}$  in terms of the Lagrange multiplier  $\Phi_{IJKL}$  and the  $B$  field, it is clear that  $\Phi_{IJKL}$  and  $B^{IJ}$  must satisfy some restrictions coming from the fulfilling of the Bianchi identities. In addition, the substitution of the solutions given in Eq. (7) into Eq. (5) will give two different expressions for the curvature in terms of the tetrad field  $e^I$ . Let us consider each case separately.

### A. Case $B^{IJ} = \kappa_1 * (e^I \wedge e^J)$

Substituting the two-form (7a) into the Eq. (5) the components of the curvature take the form

$$F_{IJKL} = \frac{2\gamma^2\kappa_1}{\gamma^2 - \sigma} \left[ \Phi_{IJKL}^* - \frac{1}{\gamma} * \Phi_{IJKL}^* + \left( \frac{l_2\sigma}{\gamma} - l_1 \right) \varepsilon_{IJKL} + 2\sigma \left( \frac{l_1}{\gamma} - l_2 \right) \eta_{[I|K|} \eta_{J]L} \right]. \quad (9)$$

Inserting (9) into the Bianchi identities (8), we get the next relations among the components of the field  $\Phi_{IJKL}$

$$\Phi_{IJKL}^* + \Phi_{IKLJ}^* + \Phi_{ILJK}^* - \frac{1}{\gamma} (*\Phi_{IJKL}^* + *\Phi_{IKLJ}^* + *\Phi_{ILJK}^*) + 3 \left( \frac{l_2\sigma}{\gamma} - l_1 \right) \varepsilon_{IJKL} = 0. \quad (10)$$

According to Ref. 16, the next step is to introduce (9) into the Einstein's equations with cosmological constant (4) and use the restrictions (10) in order to check whether the Einstein's equations with cosmological constant are satisfied or not. It is obtained that the solely fulfilling of the Bianchi identities implies the fulfilling of (4), except by one equation given by

$$\Phi_{IJ}{}^{IJ} - \frac{\gamma\sigma}{2} \Phi_{IJKL} \varepsilon^{IJKL} = -\frac{2\Lambda\sigma}{\kappa_1} \left( \frac{\gamma^2 - \sigma}{\gamma} \right) + 12(l_1 - l_2\gamma), \quad (11)$$

whose left-hand side involves a linear combination of the two Lorentz invariants  $\Phi^{IJ}{}_{IJ}$  and  $\Phi_{IJKL} \varepsilon^{IJKL}$ . Therefore, the remaining task is to be sure that (11) comes effectively from (3). To get this goal, we note that other equation that relates the two Lorentz invariants  $\Phi^{IJ}{}_{IJ}$  and  $\Phi_{IJKL} \varepsilon^{IJKL}$  comes from the Bianchi identities and is obtained by contracting (10) with  $\varepsilon^{IJKL}$ , which leads to

$$\Phi_{IJ}{}^{IJ} - \frac{1}{2\gamma} \Phi_{IJKL} \varepsilon^{IJKL} = -12 \left( \frac{l_2\sigma}{\gamma} - l_1 \right). \quad (12)$$

However, (12) is not enough to satisfy (11). Nevertheless, the combination of (12) with the equation of motion (3d) yields to

$$\Phi_{IJ}{}^{IJ} - \frac{\gamma\sigma}{2} \Phi_{IJKL} \varepsilon^{IJKL} = -12 \left( \frac{l_2\sigma}{\gamma} - l_1 \right) - \frac{\lambda\sigma}{2} \left( \frac{\gamma^2 - \sigma}{\gamma} \right). \quad (13)$$

Comparing (11) with (13) we see that their right-hand-sides are equal to each other provided that

$$\lambda = \frac{4\Lambda}{\kappa_1} + 4!l_2\sigma. \quad (14)$$

This means that the equations of motion obtained from the action principle (2) with the value of  $\lambda$  given in (14), imply that the Einstein's equations with cosmological constant are completely satisfied when Eq. (3c) is solved by (7a). Note that (14) relates the constant  $\kappa_1$  of the solution (7a) with the value of  $\lambda$  in the action principle.

Alternatively, it is possible to write the action principle (2) in the usual and equivalent form given by Holst's action for general relativity with cosmological constant

$$S[e, A] = \kappa_1 \int_{\mathcal{M}^4} \left\{ \left[ * (e^I \wedge e^J) + \frac{\sigma}{\gamma} e^I \wedge e^J \right] \wedge F_{IJ}[A] - \frac{\Lambda}{12} \varepsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L \right\}, \quad (15)$$

which is obtained by substituting the expression (7a) into (2) together with the value for  $\lambda$  given in (14) and the value of  $\mu$  obtained from (3c).

### B. Case $B^{IJ} = \kappa_2 e^I \wedge e^J$

In this case the introduction of (7b) into Eq. (5) gives for the curvature

$$F_{IJKL} = \frac{2\gamma^2 \kappa_2}{\gamma^2 - \sigma} \left[ \Phi_{IJKL} - \frac{1}{\gamma} * \Phi_{IJKL} + 2 \left( \frac{l_2 \sigma}{\gamma} - l_1 \right) \eta_{[I|K|} \eta_{J]L} + \left( \frac{l_1}{\gamma} - l_2 \right) \varepsilon_{IJKL} \right]. \quad (16)$$

Once again, by inserting (16) into the Bianchi identities (8) yields to the restrictions on the  $\Phi_{IJKL}$  field

$$\Phi_{IJKL} + \Phi_{IKLJ} + \Phi_{ILJK} - \frac{1}{\gamma} (* \Phi_{IJKL} + * \Phi_{IKLJ} + * \Phi_{ILJK}) + 3 \left( \frac{l_1}{\gamma} - l_2 \right) \varepsilon_{IJKL} = 0. \quad (17)$$

The contraction of (17) with  $\varepsilon^{IJKL}$  gives the equation for the two Lorentz invariants

$$\Phi_{IJ}{}^{IJ} - \frac{\gamma \sigma}{2} \Phi_{IJKL} \varepsilon^{IJKL} = 12 (l_1 - l_2 \gamma), \quad (18)$$

which can be combined with the equation of motion (3d) to give

$$\Phi_{IJ}{}^{IJ} - \frac{1}{2\gamma} \Phi_{IJKL} \varepsilon^{IJKL} = 12 (l_1 - l_2 \gamma) + \frac{\lambda \sigma}{2} \left( \frac{\gamma^2 - \sigma}{\gamma} \right). \quad (19)$$

On the other hand, using the form of the curvature given in (16) and the restrictions (17) in the Einstein's equations (4) it is concluded that they all are automatically satisfied, except by the equation given by

$$\Phi_{IJ}{}^{IJ} - \frac{1}{2\gamma} \Phi_{IJKL} \varepsilon^{IJKL} = \frac{2\Lambda}{\kappa_2} \left( \frac{\gamma^2 - \sigma}{\gamma^2} \right) - 12 \left( \frac{l_2 \sigma}{\gamma} - l_1 \right). \quad (20)$$

Comparing the right-hand-sides of (19) and (20) we fix the value for  $\lambda$  to

$$\lambda = \sigma \left( \frac{4\Lambda}{\kappa_2 \gamma} + 4! l_2 \right). \quad (21)$$

As it happens in Sec. II A, this means that the equations of motion obtained from the action principle (2) with the value of  $\lambda$  given in (21) imply that the Einstein's equations with cosmological constant are totally satisfied when Eq. (3c) is solved by (7b). Note that (21) relates the constant  $\kappa_2$  of the solution with the value of  $\lambda$  in the action principle.

Again, if the expression (7b) is substituted into (2) together with the value for  $\lambda$  given in (21) and the value of  $\mu$  obtained from (3c), it is possible to write the action principle (2) in terms of the tetrad field and a Lorentz connection as

$$S[e, A] = \frac{\kappa_2}{\gamma} \int_{\mathcal{M}^4} \left\{ \left[ * (e^I \wedge e^J) + \gamma e^I \wedge e^J \right] \wedge F_{IJ}[A] - \frac{\Lambda}{12} \varepsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L \right\}, \quad (22)$$

which is of the form of the Holst action, as the one obtained in section II A, but with a different expression for the Immirzi parameter (confront with Eq. (15)).

It should be noticed that there is a subtle difference between the two cases A and B previously discussed. Even though in the two cases we have the coupling of the cosmological constant, in the case B Newton's constant involves a  $\gamma$  factor.

We conclude by remarking that the constant  $l_1$  does not appear in any of the two values obtained for  $\lambda$  in Eqs. (14) and (21). This is because even though in the action principle (2) the two allowed volume terms [16, 21],  $l_1 B_{IJ} \wedge B^{IJ}$  and  $l_2 B_{IJ} \wedge * B^{IJ}$ , are included, from Eq. (3c) it can be seen that  $B_{IJ} \wedge B^{IJ} = 0$  for any solution of the  $B$ 's, thus when the action (2) is written in terms of the tetrad field, the term  $l_1 B_{IJ} \wedge B^{IJ}$  identically vanishes. Nevertheless the inclusion of such a term into the action principle (2) does affect the value of the Lorentz invariant  $\Phi_{IJ}{}^{IJ}$  (see Eqs. (13) and (19)).

### III. COUPLING THE SCALAR FIELD

Continuing with the analysis now we consider the coupling of a scalar field. The action principle (1) (or the one considered in Ref. 12) is quadratic in the  $B$  fields. Therefore, it is natural to keep in the action principle a polynomial dependence on the  $B$ 's when the coupling of a scalar field  $\phi$  is done. This can be achieved by introducing auxiliary fields  $\pi^\mu$  [22] in the form given by

$$S[B, A, \Phi, \mu, \pi, \phi] = S_{GR}[B, A, \Phi, \mu] + \int_{\mathcal{M}^4} \left[ a (B_{IJ} \wedge {}^* B^{IJ}) \pi^\mu \partial_\mu \phi + \left( \alpha_1 \tilde{H}_{\mu\nu} + \alpha_2 \tilde{G}_{\mu\nu} \right) \pi^\mu \pi^\nu d^4 x \right], \quad (23)$$

where  $a$ ,  $\alpha_1$  and  $\alpha_2$  are constants and  $\tilde{H}_{\mu\nu}$  and  $\tilde{G}_{\mu\nu}$  are Urbantke metrics [23] of weight one given by

$$\tilde{H}_{\mu\nu} := \frac{1}{12} \tilde{\eta}^{\alpha\beta\gamma\delta} B_{\mu\alpha}^{IJ} B_{\beta\gamma}^{KL} B_{\delta\nu}^{MN} \eta_{JK} \eta_{LM} \eta_{NI}, \quad (24a)$$

$$\tilde{G}_{\mu\nu} := \frac{1}{3} \tilde{\eta}^{\alpha\beta\gamma\delta} B_{\mu\alpha}^{IJ} B_{\beta\gamma}^{KL} B_{\delta\nu}^{MN} \eta_{IN} \varepsilon_{JKLM}, \quad (24b)$$

where  $B^{IJ} = \frac{1}{2} B_{\alpha\beta}^{IJ} dx^\alpha \wedge dx^\beta$ . Here  $\tilde{\eta}^{\alpha\beta\gamma\delta}$  is such that  $\tilde{\eta}^{0123} = 1$ ;  $\alpha, \beta, \gamma, \dots = 0, 1, 2, 3$  are spacetime indices,  $d^4 x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ , and  $\varepsilon_{0123} = \epsilon$  equal to  $+1$  or  $-1$  depending on the orientation chosen.

The definition of the two metrics (24) is inspired by the two Urbantke metrics introduced in Ref. 12. Even though the metrics given in Eqs. (24) are very closed to those reported in Ref. 12, there is a slight difference between them: the ones reported here are covariant metrics of weight one whereas the others are contravariant metrics of weight two<sup>2</sup>. Notice also that in Ref. 12 were considered linear combinations of the metrics introduced therein. Similarly, the Urbantke metrics used in Ref. 24 are specific linear combinations of the metrics (24).

The variation of this action with respect to the independent fields gives the equations of motion

$$\begin{aligned} \delta B : \left\{ (F_{IJKL} + \frac{1}{\gamma} {}^* F_{IJKL}) e_\gamma^K e_\delta^L - \Phi_{IJKL} B_{\gamma\delta}^{KL} + a \varepsilon_{IJKL} B_{\gamma\delta}^{KL} \phi^\mu \partial_\mu \phi \right. \\ \left. + \frac{1}{3} B_{\phi\gamma}^{KL} B_{\delta\theta}^{MN} \pi^\phi \pi^\theta \eta_{KN} (\alpha_1 \eta_{IL} \eta_{JM} + 4\alpha_2 \varepsilon_{IJLM}) \right\} \tilde{\eta}^{\alpha\beta\gamma\delta} \\ + \frac{2}{3} B_{\phi\gamma}^{KL} B_{\delta\theta}^{MN} \pi^\alpha \pi^\theta \eta_{IN} (\alpha_1 \eta_{JK} \eta_{LM} + 4\alpha_2 \varepsilon_{JKLM}) \tilde{\eta}^{\beta\phi\gamma\delta} = 0, \end{aligned} \quad (25a)$$

$$\delta A : DB^{IJ} + \frac{1}{\gamma} D {}^* B^{IJ} = 0, \quad (25b)$$

$$\delta \Phi : B^{IJ} \wedge B^{KL} + 2\mu \varepsilon^{IJKL} = 0, \quad (25c)$$

$$\delta \mu : \Phi_{IJKL} \varepsilon^{IJKL} = 0, \quad (25d)$$

$$\delta \pi : (B_{IJ} \wedge {}^* B^{IJ}) \partial_\mu \phi + 2 \left( \alpha_1 \tilde{H}_{\mu\nu} + \alpha_2 \tilde{G}_{\mu\nu} \right) \pi^\nu d^4 x = 0, \quad (25e)$$

$$\delta \phi : \partial_\mu ({}^* B_{\alpha\beta}^{IJ} B_{IJ\gamma\delta} \pi^\mu \tilde{\eta}^{\alpha\beta\gamma\delta}) = 0. \quad (25f)$$

Note that the equations of motion given in (25b) and (25c) are the same ones obtained in Sec. II (see Eqs. (3b) and (3c)). This means that the  $B$  field has the expressions given in (7) and that there is no torsion, so the curvature  $F^{IJ}$  satisfies the Bianchi identities (8). In Sec. II we used the equations of motion obtained from the proposed action principle (2) in order to show that they imply the Einstein's equation with the cosmological constant. Additionally, it was also shown that the Holst's action with the cosmological constant is obtained once the action principle (2) is written in terms of the tetrad field. In this section we will focus our attention in rewriting the action principle (23) in terms of the tetrads and the scalar field. In order to do this we need to solve (25e) for the auxiliary field  $\pi$  in terms of the tetrad and the scalar field.

The substitution of the solutions given in (7) into (24) and (25e) will lead to two different expressions for the field  $\pi^\mu$  in terms of the tetrad field  $e^I$ . Let us analyze each case separately.

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<sup>2</sup> It is also possible to define contravariant metrics of weight three given by  $\tilde{B}^{\mu\alpha IJ} \tilde{B}_{\alpha\beta}^{KL} \tilde{B}^{\beta\nu MN} \eta_{IN} \varepsilon_{JKLM}$  and  $\tilde{B}^{\mu\alpha IJ} \tilde{B}_{\alpha\beta}^{KL} \tilde{B}^{\beta\nu MN} \eta_{IN} \eta_{JK} \eta_{LM}$ . These metrics and the ones of Ref. 12 could also be used to make the coupling of scalar field to general relativity or to other theories of gravity by changing the tensorial nature and weight of the auxiliary fields involved.

**A. Case  $B^{IJ} = \kappa_1 * (e^I \wedge e^J)$**

Inserting  $B_{\alpha\beta}^{IJ} = \kappa_1 \varepsilon^{IJ}{}_{KL} e_\alpha^K e_\beta^L$  into Eq. (24) yields to

$$\tilde{H}_{\mu\nu} = \kappa_1^3 \sigma \epsilon \det(e_\alpha^I) g_{\mu\nu}[e], \quad (26a)$$

$$\tilde{G}_{\mu\nu} = 0. \quad (26b)$$

where  $g_{\mu\nu}[e] := e_\mu^I e_\nu^J \eta_{IJ}$ .

From (7a), (25e), and (26), we get

$$\pi^\alpha = -\frac{6a}{\alpha_1 \kappa_1} g^{\alpha\beta}[e] \partial_\beta \phi, \quad (27)$$

with  $g^{\alpha\beta}[e] g_{\beta\gamma}[e] = \delta_\gamma^\alpha$ .

Using (7a), (26), (27), and  $\det(e_\alpha^I) = \sqrt{\sigma g}$  with  $g := \det(g_{\mu\nu}[e]) = (\det(e_\alpha^I))^2 \sigma$ , it is possible to rewrite the proposed action principle (23) as

$$S[e, A, \phi] = \kappa_1 \int_{\mathcal{M}^4} \left\{ \left[ * (e^I \wedge e^J) + \frac{\sigma}{\gamma} e^I \wedge e^J \right] \wedge F_{IJ}[A] - 8\pi G \sqrt{\sigma g} g^{\mu\nu}[e] \partial_\mu \phi \partial_\nu \phi d^4 x \right\}, \quad (28)$$

where it was considered  $\frac{36a^2}{\alpha_1} \sigma \epsilon = 8\pi G$  with  $G$  the Newton constant. Now, it is clear that  $\frac{1}{16\pi G}$  times Eq. (28) is the wanted action in the first-order formalism, i.e., we have shown that Eq. (23) is equivalent to Holst's action for general relativity coupled to a scalar field.

**B. Case  $B^{IJ} = \kappa_2 e^I \wedge e^J$**

When the two-form  $B$  takes the form given in Eq. (7b), the Urbantke metrics (24) acquire the expressions

$$\tilde{H}_{\mu\nu} = 0, \quad (29a)$$

$$\tilde{G}_{\mu\nu} = \kappa_2^3 \epsilon \det(e_\alpha^I) g_{\mu\nu}[e]. \quad (29b)$$

By plugging Eqs. (7b) and (29) into (25e) it is obtained that the field  $\pi^\mu$  takes the form

$$\pi^\alpha = -\frac{6a}{\alpha_2 \kappa_2} g^{\alpha\beta}[e] \partial_\beta \phi. \quad (30)$$

So, in this case, the action (23) in terms of the tetrad field is given by

$$S[e, A, \phi] = \frac{\kappa_2}{\gamma} \int_{\mathcal{M}^4} \left\{ \left[ * (e^I \wedge e^J) + \gamma e^I \wedge e^J \right] \wedge F_{IJ}[A] - 8\pi G \epsilon \sqrt{\sigma g} g^{\mu\nu}[e] \partial_\mu \phi \partial_\nu \phi d^4 x \right\}, \quad (31)$$

where it was used the relation  $\frac{36a^2}{\alpha_2} \gamma \epsilon = 8\pi G$ .

Some remarks follow: (i) By using the results obtained in Secs. III A and III B, it is possible to fix the relationship among the constants  $a$ ,  $\alpha_1$ , and  $\alpha_2$ , as  $\frac{\alpha_2}{\alpha_1} = \sigma \gamma$  and  $\alpha_2 = \frac{9a^2 \gamma \epsilon}{2\pi G}$ . Thus, we might define the  $BF$  action principle for gravity with scalar field as

$$\begin{aligned} S[B, A, \Phi, \mu, \pi, \phi] &= S_{GR}[B, A, \phi, \mu] \\ &+ \int_{\mathcal{M}^4} \left[ a (B_{IJ} \wedge * B^{IJ}) \pi^\mu \partial_\mu \phi + \frac{9a^2}{2\pi G} (\gamma \tilde{G}_{\mu\nu} + \sigma \tilde{H}_{\mu\nu}) \pi^\mu \pi^\nu d^4 x \right]. \end{aligned} \quad (32)$$

As in the cases A and B, the equations of motion obtained from the variation of the action (32) with respect to the independent fields give the equations of general relativity with scalar field. Note that  $a$  can be absorbed by redefining  $\pi^\mu$ . (ii) Notice that we could have added the term  $(B_{IJ} \wedge B^{IJ}) \pi^\mu \partial_\mu \phi$  when we began the coupling of a scalar field in the action principle (23). Nevertheless, the inclusion of that term would have followed a behavior similar to the one described in the last paragraph of Sec. II for the cosmological constant. (iii) We have shown that the action for gravity coupled to a scalar field in the first-order formalism has arisen from the proposed  $BF$  action (23). Therefore, the action (23) is a good action to describe the scalar field coupling. Alternatively, the same conclusion can be reached by handling the equations of motion coming from the action principle (23), i.e., the first four equations in (25) become Einstein's equations coupled to a scalar field whereas Eq. (25f) becomes the Klein-Gordon equation once the auxiliary fields (27) or (30) are substituted into them. (iv) Notice that the difference between the cases A and B previously found for the cosmological constant is also present in the cases of the coupling of the scalar field.

#### IV. COUPLING THE YANG-MILLS FIELD

In order to couple Yang-Mills fields to the action (1) we consider

$$S[B, A, \Phi, \mu, \mathcal{A}, \phi] = S_{GR}[B, A, \Phi, \mu] + \int_{\mathcal{M}^4} K_{ab} \left( a \mathcal{F}^a[A] \wedge \phi_{IJ}^b B^{IJ} + b \mathcal{F}^a[A] \wedge \phi_{IJ}^b * B^{IJ} - \frac{\beta_1}{2} \phi_{IJ}^a \phi_{KL}^b B^{IJ} \wedge B^{KL} - \frac{\beta_2}{2} \phi_{IJ}^a \phi_{KL}^b B^{IJ} \wedge * B^{KL} \right), \quad (33)$$

where  $a$ ,  $b$ ,  $\beta_1$  and  $\beta_2$  are constants. The Yang-Mills field  $\mathcal{A} = \mathcal{A}^a J_a$  and the auxiliary field  $\phi_{IJ} = \phi_{IJ}^a J_a$ , with  $[J_a, J_b] = f^c{}_{ab} J_c$ , take values in the Lie algebra of the gauge group and  $K_{ab}$  is its Killing-Cartan metric. The variation of the action (33) with respect to the independent fields gives

$$\delta B : F_{IJ} + \frac{1}{\gamma} * F_{IJ} - \Phi_{IJKL} B^{KL} + K_{ab} \left( a \mathcal{F}^a \wedge \phi_{IJ}^b + b \mathcal{F}^a \wedge * \phi_{IJ}^b - \beta_1 \phi_{IJ}^a \phi_{KL}^b B^{KL} - \frac{\beta_2}{2} \phi_{IJ}^a * \phi_{KL}^b B^{KL} - \frac{\beta_2}{2} \phi_{KL}^a * \phi_{IJ}^b B^{KL} \right) = 0, \quad (34a)$$

$$\delta A : DB^{IJ} + \frac{1}{\gamma} D * B^{IJ} = 0, \quad (34b)$$

$$\delta \Phi : B^{IJ} \wedge B^{KL} + 2\mu \varepsilon^{IJKL} = 0, \quad (34c)$$

$$\delta \mu : \Phi_{IJKL} \varepsilon^{IJKL} = 0, \quad (34d)$$

$$\delta \mathcal{A}^a : \mathcal{D}(\phi_{aIJ} B^{IJ}) = 0, \quad (34e)$$

$$\delta \phi^a : K_{ab} \left( a \mathcal{F}^a \wedge B^{IJ} + b \mathcal{F}^a \wedge * B^{IJ} - \beta_1 \phi_{KL}^a B^{IJ} \wedge B^{KL} - \frac{\beta_2}{2} \phi_{KL}^a B^{KL} \wedge * B^{IJ} - \frac{\beta_2}{2} \phi_{KL}^a B^{IJ} \wedge * B^{KL} \right) = 0, \quad (34f)$$

where, it was used the notation  $\mathcal{F}^a$  for  $\mathcal{F}^a[A] = d\mathcal{A}^a + \frac{1}{2} f^a{}_{bc} \mathcal{A}^b \wedge \mathcal{A}^c$ , and the definition  $\mathcal{D}u^a = du^a + f^a{}_{bc} \mathcal{A}^b \wedge u^c$  for  $u = u^a J_a$ .

Notice that, since Eqs. (34b) and (34c) are equal to (25b) and (25c), the modified action (33) keeps two basic properties:  $A$  is the spin-connection (thus its curvature satisfies (8)) and the two-form  $B$ 's can be written in terms of the tetrad field as given in (7). We are going to show that the proposed action (33) is equivalent to Holst's action for gravity coupled to the Yang-Mills field. As it should be clear to the reader by now, the simplest way to obtain the goal is to solve the Eq. (34f) for the auxiliary fields. Again, it is convenient to consider each case of (7) separately.

##### A. Case $B^{IJ} = \kappa_1 * (e^I \wedge e^J)$

Taking for  $B^{IJ}$  the expression given in (7a) and assuming  $\det(e_\alpha^I) \neq 0$  it is possible to solve (34f) for  $\phi_{IJ}^a$  as

$$\phi_{IJ}^a = \frac{\sigma}{2\kappa_1} \left( \frac{1}{\beta_2^2 - \beta_1^2 \sigma} \right) [(a\beta_2 - b\beta_1) \mathcal{F}_{IJ}^a + (b\beta_2 - \sigma a\beta_1) * \mathcal{F}_{IJ}^a], \quad (35)$$

with  $* \mathcal{F}_{IJ}^a = \frac{1}{2} \varepsilon_{IJ}{}^{KL} \mathcal{F}_{KL}^a$ . By plugging this expression and the solution (7a) into the action (33), and using the Bianchi identities (8), the action (33) becomes

$$S[e, \mathcal{A}] = \kappa_1 \int_{\mathcal{M}^4} \left[ \epsilon \det(e_\alpha^I) F^{IJ}{}_{IJ} [A[e]] d^4x + \frac{K_{ab}}{2\kappa_1} \left( \frac{1}{\beta_2^2 - \beta_1^2 \sigma} \right) \left\{ [2ab\beta_2 - (a^2\sigma + b^2)\beta_1] \mathcal{F}^a \wedge \mathcal{F}^b + [(a^2\sigma + b^2)\beta_2 - 2ab\beta_1\sigma] \mathcal{F}^a \wedge * \mathcal{F}^b \right\} \right], \quad (36)$$

which is the usual action that describes the coupling of the Yang-Mills field to general relativity in the first-order formalism supplemented with the Pontrjagin term.

### B. Case $B^{IJ} = \kappa_2 e^I \wedge e^J$

Following an analogous procedure to the case A, i.e. considering (7b) and assuming  $\det(e_\alpha^I) \neq 0$ , Eq. (34f) can be solved for  $\phi_{IJ}^a$  as

$$\phi_{IJ}^a = \frac{\sigma}{2\kappa_2} \left( \frac{1}{\beta_2^2 - \beta_1^2 \sigma} \right) [(b\beta_2\sigma - a\beta_1)\mathcal{F}_{IJ}^a + (a\beta_2 - b\beta_1)*\mathcal{F}_{IJ}^a], \quad (37)$$

and by using this solution the action (33) acquires the form

$$S[e, \mathcal{A}] = \frac{\kappa_2}{\gamma} \int_{\mathcal{M}^4} [\epsilon \det(e_\alpha^I) F^{IJ}{}_{IJ} [A[e]] d^4x + \frac{\gamma \sigma K_{ab}}{2\kappa_2} \left( \frac{1}{\beta_2^2 - \beta_1^2 \sigma} \right) ([2ab\beta_2 - (a^2\sigma + b^2)\beta_1]\mathcal{F}^a \wedge \mathcal{F}^b + [(a^2\sigma + b^2)\beta_2 - 2ab\beta_1\sigma]\mathcal{F}^a \wedge *\mathcal{F}^b)] \quad (38)$$

We conclude this section by making some remarks: (i) Note that the coefficients of the Pontrjagin and Yang-Mills terms that appear in the final action principles (36) and (38) depend on the whole set of coefficients of the terms added to the original action principle in (33). Observe also that different combinations of those terms allow us to make the coupling of the Yang-Mills field into the action principle (1). (ii) Notice that a term of the form  $\frac{\beta_3}{2} \phi_{IJ}^a \phi_{KL}^b *B^{IJ} \wedge *B^{KL}$  can be added to the action principle (33). If such a term is included its effect would be a redefinition of the  $\beta_1$  parameter. (iii) Finally, note that the coupling of the Yang-Mills field is also useful to obtain the coupling of the Maxwell field by doing the replacement  $\phi_{IJ}^a \rightarrow \phi_{IJ}$  in the auxiliary fields.

## V. CONCLUDING REMARKS

We have done the coupling of the cosmological constant, the scalar, Yang-Mills, and Maxwell fields to general relativity in the theoretical framework of  $BF$  gravity with Immirzi parameter. The analysis was carried out using the action principle for gravity used in Ref. 13. Our analysis shows that the coupling can be done without any technical or conceptual difficulties. Therefore, the framework of  $BF$  gravity with Immirzi parameter is robust enough to allow the coupling of matter fields. The action principles for the matter fields include all the terms that *a priori* might contribute to the coupling. The role that each one of these terms plays can be clearly appreciated when the action principles are written in the first-order and second-order formalisms. This fact also allows us to see that the proposed action principles for the coupling of the matter fields are the right ones. From these action principles, Einstein's equations and matter field equations follow immediately. This was explicitly shown for the coupling of the cosmological constant. For the scalar field, Maxwell, and Yang-Mills field it was shown that the proposed action principles are equivalent to the usual action principles in the first-order formalism.

Notice that the results presented in this work can be used to obtain the coupling of matter fields to the CMPR action principle by performing the transformation introduced in Ref. 14. It could be interesting to compare the resulting couplings with the ones obtained directly from the CMPR action principle (see Refs. 16 and 25). For example, in the case of the cosmological constant these two approaches agree as can be seen from the comparison of the results presented in the Sec. II and the ones presented in the Sec. 4 of Ref. 14 where the coupling of the cosmological constant to the action principle (1) is obtained by doing a transformation from the CMPR action principle.

We think that our approach and results are relevant because they display the way matter fields couple to general relativity written as a constrained  $BF$  theory. The couplings are restricted by the geometric and tensor nature of the  $B$  fields, the matter fields and the auxiliary fields themselves. Therefore, when all these fields are put together, they lead to the coupling terms introduced in this paper. The simultaneous coupling of all the matter fields is immediate. Even though our results are classical they might be useful or interesting for the spinfoam approach to quantizing gravity because it is based on a reformulation of general relativity as a constrained  $BF$  theory [2].

Our results are also relevant for the modification of general relativity recently proposed by Krasnov [26–28] (and discussed in Refs. 17, 24, 29, 30, 31, and 32) aiming to developing a renormalizable theory of gravity from  $BF$  theory. This is so because the matter coupling terms introduced in this paper can be used to make the matter couplings in the framework of such theories.

The coupling of fermion fields of spin  $\frac{1}{2}$ ,  $\frac{3}{2}$ , etc. to  $BF$  gravity was not analyzed in this paper. The reason is that such a coupling deserves a separate treatment because of the various types of fermions and also because of the various ways fermions couple to gravity, but such an analysis must be done and confronted to the classical and semiclassical limits of the quantum theory developed in Ref. 33. This issue as well as the study of supersymmetric fields and exotic matter is left for future work.



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